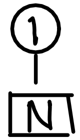
Ex (contd)

$G = \mathbb{C}^\times$, $N = \mathbb{C}^{\oplus N}_{wt\,1}$



- $\mathcal{M}_C = \{ xy = w^N \}$
- $\mathcal{M}_H = N \oplus N^* /\!/\!/ G = \left\{ \begin{array}{c} \mathbb{C} \\ {\scriptstyle b}\downarrow\uparrow{\scriptstyle a} \\ \mathbb{C}^N \end{array} \;\middle|\; ab = 0 \right\} /\!/ \mathbb{C}^\times \cong \overline{\mathbb{O}}_{\mathfrak{sl}_N, \min}$

  minimal nilpotent orbit for $\mathfrak{sl}_N$

Unless $N=2$, they are different varieties. We will see their mysterious relations (called symplectic duality) later.

no conceptual understanding yet.
collection of interesting phenomena

Take symplectic resolution $\widetilde{\mathcal{M}}_C$, $\widetilde{\mathcal{M}}_H$

$\widetilde{\mathcal{M}}_C =$

$\mathbb{P}^1$'s intersect as A_{N-1} Dynkin diagram

$\widetilde{\mathcal{M}}_H = T^*\mathbb{P}^{N-1}$

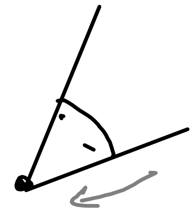
Observe

$e(\widetilde{\mathcal{M}}_C) = N = e(\widetilde{\mathcal{M}}_H)$

§3. torus action

(1) $H_*^{G(\mathcal{O})}(\mathcal{R})$ has the homological degree

$\Longrightarrow \quad \mathbb{C}^\times \curvearrowright \mathcal{M}_C$, but it is not necessarily conical



as it is not $\mathbb{Z}_{\geq 0}$-degree in general.

<u>ex.</u> $\mathcal{M}_C(\mathbb{C}^\times, N=0) = \mathbb{C} \times \mathbb{C}^\times \curvearrowleft \mathbb{C}^\times$ mult. on the 1st factor

(2) $\pi_0(\mathcal{R}) = \pi_0(Gr_G) = \pi_1(G) \quad \therefore \quad H_*^{G(\mathcal{O})}(\mathcal{R})$ is $\pi_1(G)$-graded.

$\therefore \quad \underset{\text{Pontryagin dual}}{\pi_1(G)^\wedge} \curvearrowright \mathcal{M}_C$ e.g. $G = GL_n \Rightarrow \pi_1(G) = \mathbb{Z} \Rightarrow \pi_1(G)^\wedge = \mathbb{C}^\times$

<u>Ex.</u> $\mathbb{C} \times \mathbb{C}^\times \curvearrowleft \mathbb{C}^\times$: mult. on the 2nd factor

<u>Rem.</u> Poisson bracket is preserved.

<u>Remark</u> (symplectic duality)

$\mathrm{Hom}_{grp}(\mathbb{C}^\times, \pi_1(G)^\wedge) \cong \mathrm{Hom}_{grp}(G, \mathbb{C}^\times)$

$\hookleftarrow$ appears for GIT quotient

$M /\!\!/\!\!/_{\chi} G = \mathcal{M}_H^\chi$

$\mathbb{C}^\times$-action on $\mathcal{M}_C$ $\longleftrightarrow$ stability condition on $\mathcal{M}_H$ (Higgs branch)

§4. Flavor symmetry

Suppose N is a representation of a larger group $\tilde{G}$ containing G as a normal subgroup.

$$G_F := \tilde{G}/G \quad \text{flavor symmetry}$$

$\tilde{G}(\mathcal{O}) \curvearrowright \mathrm{Gr}_G, \mathcal{T}, \mathcal{R}$ etc

$\Longrightarrow$ $\operatorname{Spec} H_*^{\tilde{G}(\mathcal{O})}(\mathcal{R})$: deformation of $\mathcal{M}_C$ parametrized by $\operatorname{Spec} H^*_{G_F}(pt) = \mathfrak{t}_F/W_F$

Assume $G_F = T_F$ torus

Consider $\mathcal{R} = \mathcal{R}_{\tilde{G},N}$ for $(\tilde{G}, N)$.

$$\mathcal{M}_C(\tilde{G}, N) = \operatorname{Spec} H_*^{\tilde{G}(\mathcal{O})}(\mathcal{R}_{\tilde{G},N}) \curvearrowleft \pi_1(\tilde{G})^\wedge \longleftarrow \pi_1(T_F)^\wedge = T_F^\vee$$

<u>Prop.</u> $\mathcal{M}_C(G,N) \cong \mathcal{M}_C(\tilde{G},N) /\!\!/\!\!/ T_F^\vee$ (symplectic reduction)

One can also consider GIT quotient

$$\mathcal{M}_C(\tilde{G},N) /\!\!/\!\!/_{\rho} T_F^\vee = \mu_F^{-1}(0) /\!\!/_{\rho} T_F^\vee \quad \text{for } \rho \in \operatorname{Hom}(T_F^\vee, \mathbb{C}^\times) \cong \operatorname{Hom}(\mathbb{C}^\times, T_F)$$

Rem (symplectic duality, 2^{nd} instance)

deformation / GIT quotient for $\mathcal{M}_C$ $\longleftrightarrow$ $\mathbb{C}^\times$-action on $\mathcal{M}_H = M/\!\!/\!\!/G \curvearrowleft \tilde{G}/G = T_F$

$\rho \in \mathrm{Hom}(\mathbb{C}^\times, T_F)$

Example : Goto–Bielawski–Dancer–toric hyperKähler manifold

$$1 \to T \to (\mathbb{C}^\times)^n \to T_F \to 1 \qquad \mathbb{C}^n \oplus (\mathbb{C}^n)^* /\!\!/\!\!/_{\xi}\, T \quad \text{for } \xi : T \to \mathbb{C}^\times$$

$\curvearrowright N = \mathbb{C}^n$

$$\mathcal{M}_C(T, N) = \underbrace{\mathcal{M}_C((\mathbb{C}^\times)^n, N)}_{\substack{\| \\ \mathbb{C}^n \oplus (\mathbb{C}^n)^* \\ \text{by Example 2}}} /\!\!/\!\!/ T_F^\vee$$

: toric hyperKähler manifold associated with

$$1 \to T_F^\vee \to (\mathbb{C}^\times)^n \to T^\vee \to 1$$

§5. Birational description ("classical" Coulomb branch)

Classical in the sense "not quantum".

- $H_*^{G(\mathcal{O})}(\mathcal{R}) \cong H_*^{T(\mathcal{O})}(\mathcal{R})^W$
- $H_*^{T(\mathcal{O})}(\mathcal{R}) \otimes_{H_T^*(pt)} \operatorname{Frac} H_T^*(pt) \underset{\text{localization thm}}{\cong} H_*^{T(\mathcal{O})}(\mathcal{R}^T) \otimes_{H_T^*(pt)} \operatorname{Frac} H_T^*(pt)$

- $(Gr_G)^T = Gr_T$
- $N(\mathcal{O})^T = N^T(\mathcal{O})$
  $N(K)^T = N^T(K)$

$\Rightarrow \mathcal{R}^T = \mathcal{T}^T = \mathcal{T}$ for (T, N^T)

$\therefore \operatorname{Spec} H_*^{T(\mathcal{O})}(\mathcal{R}^T) = \mathcal{M}_C(T, \underset{\text{trivial rep.}}{N^T}) \underset{\text{Example 1}}{\cong} \mathfrak{t} \times T^\vee$

$\mathfrak{t} = \operatorname{Lie} T$
$T \subset G$ max. torus

$\therefore$

$$\begin{tikzcd} \mathcal{M}_C \arrow[rr, dashed, "\text{birational"}] \arrow[dr, "\varpi"'] & & \mathfrak{t} \times T^\vee / W \arrow[dl, "1^{st}\text{ proj.}"] \\ & \mathfrak{t}/W \arrow[u, phantom, "\circlearrowright"] & \end{tikzcd}$$

"classical" Coulomb branch

$\therefore$ general fiber of ϖ $= T^\vee$

Detour $\mathbb{C}^\times_{\mathrm{loop}} \curvearrowright \mathcal{R}$ loop rotation

Q. What is $\mathcal{T}^{\mathbb{C}^\times_{\mathrm{loop}}} \longrightarrow N(K)^{\mathbb{C}^\times_{\mathrm{loop}}}$

This question is relevant for geometric analysis of representations of quantized Coulomb branches (cf. Cherednik-Ginzburg, Vasserot)
DAHA

☆ $\mathrm{Gr}_G^{\mathbb{C}^\times_{\mathrm{loop}}} \cong \coprod_{\mu:\text{ dominant cochar.}} \underbrace{G\cdot z^\mu}_{\cong G/P_\mu}$ $\left(g_0 (z\tau)^\mu = g_0 z^\mu \underbrace{\tau^\mu}_{\in G(\mathcal{O})} \right)$

☆ $N(K)^{\mathbb{C}^\times_{\mathrm{loop}}} \cong N$ $\left(\begin{array}{l} s(z) = a_n z^n + \cdots + a_0 + a_1 z + \cdots \\ s(z) = s(z\tau) \ \forall \tau \iff a_i = 0 \text{ except } i = 0 \end{array} \right)$

☆ $\left(\mathcal{T} = G(K) \times^{G(\mathcal{O})} N(\mathcal{O})\right)^{\mathbb{C}^\times_{\mathrm{loop}}} \cong \coprod_\mu G \times^{P_\mu} \underbrace{N^{\mu \leq 0}}$ direct sum of wt subspaces with $\langle \mathrm{wt}, \mu \rangle \leq 0$

Ψ

$$\left(\begin{array}{l} [\underbrace{g(z)}_{g_0 z^\mu}, s(z)] = [\underbrace{g(z\tau)}_{g_0 z^\mu \tau^\mu}, s(z\tau)] = [g_0 z^\mu, \tau^\mu s(z\tau)] \\ \text{Moreover } s(z) \in N(\mathcal{O}) \iff s(1) \in N^{\mu \leq 0} \end{array} \right.$$

$\therefore s(z) = \tau^\mu s(z\tau)$

$\therefore s(z) = z^{-\mu} \underbrace{s(1)}_{\in N}$

§6. Previously Known Examples

- Bezrukavnikov - Finkelberg - Mirković

  G : arbitrary , $N = 0$

  Langlands dual

  $$\Longrightarrow \quad \mathcal{M}_C \cong \mathfrak{Z}^{G^\vee}_{\mathfrak{g}^\vee} = \{ (g, X) \in G^\vee \times \mathfrak{g}^\vee \mid X : \text{regular} \} / \text{conjugacy}$$

  universal centraliser

  $$\cong \bar{\mu}^{-1}(y, y) / N^\vee \times N^\vee$$

  $\mathfrak{n}^\vee < \mathfrak{g}^\vee$, $N^\vee \subset G^\vee$

  $y \in \mathfrak{n}^\vee \cong \mathfrak{n}^{\vee *}$

  $\bar{\mu} : G^\vee \times \mathfrak{g}^\vee \xrightarrow{\mu} \mathfrak{g}^{\vee *} \times \mathfrak{g}^{\vee *} \longrightarrow \mathfrak{n}^{\vee *} \times \mathfrak{n}^{\vee *}$

  $g, X \longmapsto (X, -\mathrm{Ad}_{g^{-1}}(X))$

  quantization = quantum hamiltonian reduction = Toda lattice

- Vasserot , Varagnolo - Vasserot (quantized , flag version)

  Bezrukavnikov - Finkelberg - Mirković

  G : arbitrary , $N = \mathfrak{g}$

  $$\Longrightarrow \quad \mathcal{M}_C \cong \mathfrak{t} \times T^\vee / W$$

  ("classical" Coulomb branch)

  quantization (with flavor symmetry $\mathbb{C}^\times \curvearrowright \mathfrak{g}$)

  = spherical part of the trigonometric DAHA

Part II Quiver gauge theories

§1. Definition

$Q = (Q_0, Q_1)$: quiver $\qquad Q_1 \longrightarrow Q_0 \times Q_0$

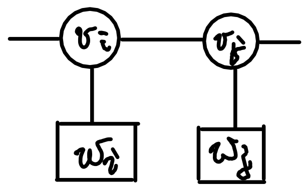
Q_0: vertexes, Q_1: edges $\qquad h \mapsto o(h) \to i(h)$

$\mathbf{V}, \mathbf{W}$: Q_0-graded finite dimensional complex vector spaces

$$\mathbb{G} := \prod_{i \in Q_0} GL(V_i), \quad \mathbb{N} := \bigoplus_{h \in Q_1} \mathrm{Hom}(V_{o(h)}, V_{i(h)}) \oplus \bigoplus_{i \in Q_0} \mathrm{Hom}(W_i, V_i)$$

(use \mathbf to avoid conflicts with groups for (another) affine Grassmann)

<u>notation</u>



$v_i = \dim V_i$

$w_i = \dim W_i$

<u>Remarks</u> (1) $\mathcal{M}_H = \mathbb{N} \oplus \mathbb{N}^* /\!/\!/_{\chi}\, \mathbb{G}$ is a quiver variety.

(2) $H^{\mathbb{G}}_*\left((\mathcal{T} \times_{\mathbb{N}(K)} \mathcal{T})^{\mathbb{C}^\times_{\mathrm{loop}}} \right) \underset{\text{Varagnolo-Vasserot}}{=}$ quiver Hecke algebra aka KLR algebra ($W=0$) and its generalization

Therefore representation theory of quantized Coulomb branches of quiver gauge theories are related with canonical bases.

<u>Def.</u> $\mathcal{M}_C \equiv \mathcal{M}_C(G, N) \equiv \mathcal{M}_C(\mathbf{V}, \mathbf{W})$ Coulomb branch

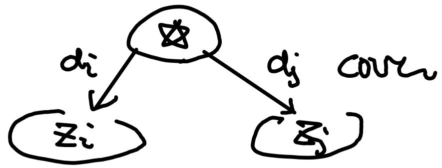
- $\pi_1(G)^\wedge = (\mathbb{C}^\times)^{Q_0} =: T^{Q_0} \curvearrowright \mathcal{M}_C$ opposite to quiver varieties
- deformation parametrized by Spec $H^*_{G_F}(pt)$ $G_F = \prod GL(W_i) \times (\mathbb{C}^\times)^{b_1(Q)}$

<u>Remark</u> (symmetrizable case)

(a_{ij}) : Cartan matrix (d_i) s.t. $d_i a_{ij} = d_j a_{ji}$

$\Longrightarrow$ introduce z_i for $i \in Q_0$ s.t. $z_i^{1/d_i} = z_j^{1/d_j}$

consider $\mathrm{Hom}(V_i, V_j)$ on ☆



$\Longrightarrow$ $\mathcal{M}_C$ is defined in the same manner

[N-Weekes]

§ 2. Based maps

Assume Q (possibly with (d_i)) is of finite type

$\mathfrak{g}$ = corresponding cpx simple Lie algebra

G = adjoint type group (nothing to do with G_ξ)

Suppose $W = 0$. Let $\alpha = \sum \dim V_i \, \alpha_i$.

simple coroot

<u>Th</u>. $\mathcal{M}_C \cong$ moduli space of based maps $\phi : \mathbb{P}^1 \longrightarrow G/B$
of degree α $\quad \phi(\infty) = B_-$

$\parallel$

$\overset{\circ}{Z}{}^\alpha$ open zastava space

(sketch of the proof)

We first construct

$$\begin{tikzcd} \overset{\circ}{Z}^{\alpha} \arrow[r, dashed, "\text{birational}"'] \arrow[dr, "\pi"'] & \mathfrak{t}\times \mathbb{P}^{\vee}/W \arrow[d, "1^{st}\text{ proj.}"] \\ & \mathfrak{t}/W \end{tikzcd} \qquad \mathfrak{t}\times \mathbb{P}^{\vee}/W = \prod_i (\mathbb{C}\times\mathbb{C}^{\times})^{\dim V_i}/\mathfrak{S}_{\dim V_i}$$

Suppose $Q =$ type A_1, hence $G/B = \mathbb{P}^1$

$$\Longrightarrow \quad \phi(z) = \frac{R(z)}{Q(z)}$$

Q : monic polynomial of degree α

P : polynomial of degree $< \alpha$

no common zero

$w_1, \cdots, w_\alpha$: zeros of Q (counted with multiplicities)

Then we define $\overset{\circ}{Z}^{\alpha} \dashrightarrow (\mathbb{C}\times\mathbb{C}^{*})^{\alpha}/\mathfrak{S}_\alpha$

$\phi(z) \longmapsto \{(w_1, R(w_1)), \cdots, (w_\alpha, R(w_\alpha))\}$ (unordered)

Moreover $\overset{\circ}{Z}^{\alpha} \xrightarrow{\pi} \mathbb{C}^{\alpha}/\mathfrak{S}_\alpha$ is defined everywhere

$\phi \mapsto \{w_1, \cdots, w_\alpha\}$

The definition for general G is similar.

We define
$$\begin{tikzcd} \mathcal{M}_C \arrow[r, dashed, "\text{birat.}"] \arrow[dr, "\varpi"'] & \mathfrak{t} \times T^\vee / W \arrow[d] \\ & \mathfrak{t}/W \end{tikzcd}$$
appropriately.

$R(w_r) \leftrightarrow$ explicit homology cycle

Then we get
$$\begin{tikzcd} \overset{\circ}{Z}{}^\alpha \arrow[r, dashed, "\text{birat.}"] \arrow[dr] & \mathcal{M}_C \arrow[d, "\varpi"] \\ & \mathfrak{t}/W \end{tikzcd}$$
$=: \Xi$ (isomorphism over $\overset{\circ}{\mathfrak{t}}/W$)

distinct

Check (A) Show that $\begin{cases} \overset{\circ}{Z}{}^\alpha : \text{normal} \\ \overset{\circ}{Z}{}^\alpha \to \mathfrak{t}/W \quad \text{flat} \end{cases}$ corresponding properties are shown for $\mathcal{M}_C$

(B) Ξ extends $\mathfrak{t}^\bullet/W$ s.t. $\mathfrak{t} \setminus \mathfrak{t}^\bullet$ has at least codim. ≥ 2

(A) is about $\overset{\circ}{Z}{}^\alpha$, nothing to do with Coulomb branches

For (B), we localize around $\xi \in \mathfrak{t}^\bullet \setminus \mathfrak{t}^\circ$ by $H_*^{T(\mathcal{O})}(\mathcal{R})_\mathfrak{p}$ $\mathfrak{p}$ = {func. vanishing at ξ}

$\mathcal{M}_C$ for G = torus or GL $\quad \longrightarrow \quad \underline{H_*^{T(\mathcal{O})}(\mathcal{R}^\xi)_\mathfrak{p}}$ // (equal to the above)

Then Ξ extends everywhere.

//

§ 3. generalised slices

Assume Q (possibly with (d_i)) is of finite type
$\mathfrak{g}$ = corresponding cpx simple Lie algebra ← as in § 2.
G = adjoint type group

$$\lambda := \sum \dim W_i \cdot \varpi_i \quad , \quad \mu = \lambda - \sum \dim V_i \cdot \alpha_i$$

fundamental coweight simple coroot

Roughly, $\mathcal{M}_C \equiv \mathcal{M}_C(\lambda, \mu)$
is a moduli space of based maps $\mathbb{P}^1 \dashrightarrow$ flag var.
with singularity at the origin.
type of the singularity is specified by λ.

Th. $\mathcal{M}_C(\lambda, \mu)$ = generalised affine Grassmannian slice $\overline{\mathcal{W}}^{\lambda}_{\mu}$

I will not give the definition of $\overline{\mathcal{W}}^{\lambda}_{\mu}$.
I will only mention their surprising properties.

(1) (Finkelberg–Mirković, Braverman–Finkelberg)

$$\overline{\mathrm{Gr}_G^\lambda} \xleftarrow{p} \overline{W}_\mu^\lambda \xrightarrow{q} Z^{-w_0(\lambda-\mu)}$$

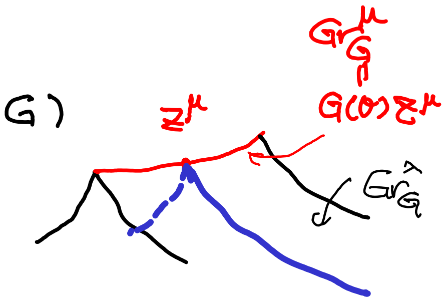
zastava space = partial compactification of moduli of based maps $\mathbb{P}^1 \to G/B$

(2) [FM, BF]

Suppose μ : dominant $\Rightarrow$ p is locally closed embedding :

$$\overline{W}_\mu^\lambda \cong \overline{\mathrm{Gr}_G^\lambda} \cap W_{G,\mu} \quad \text{where} \quad W_{G,\mu} = \underset{\underset{\mathrm{Ker}(G[z^{-1}] \xrightarrow{ev_\infty} G)}{\wr}}{G[z^{-1}]_1} \cdot z^\mu$$

affine Grassmannian slice



One show that q is birational.

$\Rightarrow$ We have an integrable system and a birational coordinate system induced from $Z^{-w_0(\lambda-\mu)}$.

▷ Check $\overline{W}_\mu^\lambda$ is correct along $\mathfrak{t}^\bullet/W$

__quantization__ (Appendix to the 2nd paper, written by BFN+ Kamnitzer, Kodera, Webster, Weekes)

shifted Yangian Y_μ

$E_i^{(q)}, F_i^{(q)}, H_i^{(p)} \quad q>0 \quad p\in\mathbb{Z}$

usual relations of Yangian $+ \begin{cases} H_i^{(p)} = 0 & p < -\langle\mu,\alpha_i\rangle \\ H_i^{(-\langle\mu,\alpha_i\rangle)} = 1 \end{cases}$

__Th__ (Weekes 1903.07734)

$$\begin{array}{ccc} Y_\mu & \longrightarrow & \text{quantized Coulomb branch}\,|_{\hbar=1} \\ \downarrow & & \wr \\ \multicolumn{3}{c}{\text{localized ring of difference operators}} \end{array}$$

($\hbar$: variable version holds if Q : finite type

Remembering relations with $H_*^{G}\left(\left(\mathcal{G}\times\mathcal{G}_{N(\mathcal{K})}\right)^{\mathbb{C}^\times_{\mathrm{loop}}}\right) \cong$ quiver Hecke algebra $\curvearrowleft$ canonical base

we have $\boxed{\text{representation theory of shifted Yangian} \longleftrightarrow \text{canonical bases}}$